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The Genesis of the Generalized Riemann Integral

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Abstract—The generalized Riemann integral with equivalent definitions spanning many years, and its superiority over the Lebesgue integral are reviewed.

Keywords—Denjoy, Generalized Riemann integral, Henstock, Kurzweil, Perron.

SECTION 1

The term "Generalized Riemann integral" refers, strictly speaking, to a certain functional as defined, independently, by Kurzweil [1] and Henstock [2,3], not to the functional itself. But there is a good reason (logically, almost a necessity) for taking this term to mean merely that functional itself. In this note, we take this approach and review equivalent definitions of that term. We start, however, with another, extremely important integral.

SECTION 2

The Lebesgue integral was introduced in [4,5]. One of its definitions cf[6, p. 213] is the following.

DEFINITION 1. *Let $-\infty < a < b < \infty$, and let f be a real function defined in $[a, b]$. Then f is said to be summable over $[a, b]$ iff there is a real function F, absolutely continuous (AC) on $[a, b]$, such that $F'(x) = f(x)$ for almost all x in $[a, b]$. In that case, the number $F(b) - F(a)$ (which is independent of the particular F chosen) is denoted $(L) \int_a^b f(x)\,dx$ and called the Lebesgue integral of f over $[a, b]$.*

We shall assume the first sentence of Definition 1 throughout this paper.

The Lebesgue integral has some very favorable properties, but it also has severe limitations. Consider the positive and negative parts of f, namely, the functions f_+, f_- defined on $[a, b]$ as follows:

$$f_+(x) = \begin{cases} f(x), & \text{if } f(x) > 0, \\ 0, & \text{otherwise;} \end{cases} \qquad f_-(x) = \begin{cases} -f(x), & \text{if } f(x) < 0, \\ 0, & \text{otherwise.} \end{cases}$$

Equivalently, throughout $[a, b]$,

$$f_+(x) = \frac{|f(x)| + f(x)}{2}, \qquad f_-(x) = \frac{|f(x)| - f(x)}{2}.$$

Now, f is summable over $[a, b]$ iff both f_+ and f_- are. Suppose f_+ and f_- are not both summable over $[a, b]$ (e.g., $|f| = f_+ + f_-$ is not summable over $[a, b]$). Then neither is f. This is a severe weakness which is not shared by an infinite series. Given an infinite series $\sum_{k=1}^{\infty}$ of real terms, consider the positive and negative parts of $a_k, k = 1, 2, \ldots$:

$$a_{k+}(x) = \begin{cases} a_k, & \text{if } a_k > 0, \\ 0, & \text{otherwise;} \end{cases} \qquad a_{k-} = \begin{cases} -a_k, & \text{if } a_k < 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\sum_{k=1}^{\infty} a_k$ can converge while $\sum_{k=1}^{\infty} a_{k+}$ and $\sum_{k=1}^{\infty} a_{k-}$ do not.

EXAMPLE. Let $a_k = (-1)^{k+1}/k$, $k = 1, 2, \dots$. Then, for $n = 1, 2, \dots$, $\sum_{k=1}^{2n-1} a_{k+} > \sum_{k=1}^{n} 1/(2k) \to \infty$, $\sum_{k=1}^{2n} a_{k-} = \sum_{k=1}^{n} 1/(2k) \to \infty$, so that $\sum_{k=1}^{\infty} a_{k+}$ and $\sum_{k=1}^{\infty} a_{k-}$ diverge while $\sum_{k=1}^{\infty} a_k$ converges (to $\log 2$).

In fact, even the improper Riemann integral does not share the above weakness. Consider, for example,

$$g(x) = x^2 \sin \frac{\pi}{x^2}, \qquad -\infty < x < \infty, \quad x \neq 0; \quad g(0) = 0.$$

Then $g'(0) = 0$, and

$$\int_{\varepsilon}^{1} g'(x)\, dx = -\varepsilon^2 \sin \frac{\pi}{\varepsilon^2} \to 0 \text{ as } \varepsilon \to 0^+,$$

so that the improper Riemann integral $(IR) \int_0^1 g'(x)\, dx$ converges while $(IR) \int_0^1 |g'(x)|\, dx$ does not, as g' is not summable over $[0, 1]$, [7, p. 135, Example; p. 115, Theorem 178(i); p. 129, Theorem 198]. Thus, $(IR) \int_0^1 g'_+(x)\, dx$ and $(IR) \int_0^1 g'_-(x)\, dx$ diverge.

Another weakness of Lebesgue integration has to do with the retrieval of a function from its derivative. We would like to have

$$(L) \int_a^b f'(x)\, dx = f(b) - f(a),$$

whenever f is (finitely) differentiable throughout $[a, b]$. This is in fact the case if f' is summable over $[a, b]$ [7, p. 183, Theorem 264]. But f may be (finitely) differentiable throughout $[a, b]$ without f' being summable over $[a, b]$, as is g (with $a = 0, b = 1$).

A third deficiency of Lebesgue integration (shared also by Riemann integration) concerns improper integration. It may happen that $(L) \int_{a+\varepsilon}^{b} f(x)\, dx$ converges as $\varepsilon \to 0^+$ without f being summable over $[a, b]$, as is the case for g', with $a = 0, b = 1$.

SECTION 3

The Generalized Riemann integral is one which is free from the above weaknesses of the Lebesgue integral. It was introduced in [8,9], but we shall give it here another definition [6, p. 241] which parallels Definition 1. It has been called also the restricted Denjoy integral. We need first two preliminary definitions.

DEFINITION 2. *Let S be a (nonempty) bounded real set. A real function F is said to be absolutely continuous in the restricted sense (AC_*) on S iff F is bounded on some closed interval*[1] $\supseteq S$ *and if, for every $\varepsilon > 0$, there is an $\eta > 0$ such that, for every finite set of nonoverlapping closed intervals whose endpoints belong to S, and whose sum of lengths is $< \eta$, the sum of oscillations of F on these intervals is $< \varepsilon$.*

On a closed interval, AC and AC_* are equivalent.

DEFINITION 3. *A real function F is said to be generalized absolutely continuous (ACG_*) on a closed interval I, iff F is continuous on I and $I = \cup_{k=1}^{\infty} S_k$ where F is AC_* on each S_k.*

Finally, here is the definition alluded to.

DEFINITION 4. *f is said to be generalized Riemann integrable (GRI) over $[a, b]$ iff there is a real function F, ACG_* on $[a, b]$, such that $F'(x) = f(x)$ for almost all x in $[a, b]$. In that case, $F(b) - F(a)$ (which is independent of the particular F chosen) is denoted $(GRI)\int_a^b f(x)\, dx$ and is called the generalized Riemann integral (GRI) of f over $[a, b]$.*

[1] Always meaning a set $[\alpha, \beta]$, $-\infty < \alpha < \beta < \infty$.

SECTION 4

f is summable over $[a,b]$ iff both f and $|f|$ are GRI over $[a,b]$, in which case

$$(L)\int_a^b f(x)\,dx = (GRI)\int_a^b f(x)\,dx.$$

A corollary is that if f is ≥ 0 or ≤ 0 on $[a,b]$, or, more generally, is bounded above or below in $[a,b]$, then its summability and its GRI over $[a,b]$ are equivalent and the corresponding integrals are equal.

GRI is free from the three defects mentioned above of summability. If (GRI)$\int_{a+\varepsilon}^b f(x)\,dx$ or (GRI)$\int_a^{b-\varepsilon} f(x)\,dx$ converges as $\varepsilon \to 0^+$, then f is GRI over $[a,b]$ and (GRI)$\int_a^b f(x)\,dx$ equals the limit. In particular, g' of Section 2 is GRI over [0,1] though not summable there. Thus, GRI is strictly stronger than summability.

Also, as g'_+ and g'_- are ≥ 0, but not summable over [0,1], they are not GRI there, though g' is.

Finally, if f is (finitely) differentiable over $[a,b]$, then f' is GRI there and

$$(GRI)\int_a^b f'(x)\,dx = f(b) - f(a).$$

SECTION 5

The following definition originated with Perron [10] (for a bounded f), was given in the present form by Bauer [11], and the fact that Definitions 4 and 5 define the same property and the corresponding integrals are equal, is due to the work of Hake [12], Alexandroff [13,14] and Looman [15]. Because of Definition 5, GRI has been called also the Perron integral.

DEFINITION 5. *f is said to be GRI over $[a,b]$ iff*

(i) *there exist real functions φ, ψ such that*

$$\varphi(a) = 0 = \psi(a)$$

and throughout $[a,b]$,

$$\limsup_{h\to 0} \frac{\varphi(x+h)-\varphi(x)}{h} \leq f(x) \leq \liminf_{h\to 0} \frac{\psi(x+h)-\psi(x)}{h},$$

($h \to 0$ should be replaced by $h \to 0^+$ for $x = a$; by $h \to 0^-$ for $x = b$).

(ii) sup $\varphi(b)$ = inf $\psi(b)$ *where the* sup(inf) *is taken over all real functions φ (ψ) satisfying the above properties involving $\varphi(\psi)$ only.*

In that case, sup $\varphi(b)$ *is denoted (GRI)$\int_a^b f(x)\,dx$ and is called the GRI of f over $[a,b]$.*

SECTION 6

The simplest definition of GRI was given much later by Kurzweil [1] and, independently, Henstock [2,3]. It is the following Definition 6 which turns out to be (as it should) equivalent to Definitions 4 and 5 (with the same value of the integral).

DEFINITION 6. *f is said to be GRI over $[a,b]$ iff there is a real number I so that, for each $\varepsilon > 0$, there is a positive function $\delta_\varepsilon(t)$ on $[a,b]$ such that*

$$\left| I - \sum_{k=1}^n f(t_k)(x_k - x_{k-1}) \right| < \varepsilon,$$

whenever $a = x_0 < x_1 < \cdots < x_n = b$ *and* $x_{k-1} \le t_k \le x_k$, $x_k - x_{k-1} < \delta_\varepsilon(t_k)$ *for* $k = 1, 2, \ldots, n$. *In that case, such an* I *is unique, is denoted (GRI)*$\int_a^b f(x)\,dx$ *and is called the GRI of* f *over* $[a, b]$.

Observe that this definition is merely a slight modification of that of Riemann integrability and Riemann integral (δ_ε being a constant).

GRI includes as special cases several other integrals (including Riemann, improper Riemann and Lebesgue). For more details see [17,18].

For a rapid study of GRI, based on Definition 6, the reader is referred to [18]. Two of the monographs on GRI are [19,20]. It seems very sensible to make GRI the standard integral of the working analyst; [21], e.g., is a text essentially doing this. (It uses, however, instead of GRI, the term gauge integrable (integral).) This usage is followed also in the monograph [22]. Also, GRI is a special case of the P-integral (the Henstock-Kurzweil integral) of the monograph [23].

SECTION 7

One application of GRI is a simple definition of Lebesgue measure, not involving any infinite series. Let S be a nonempty real set and suppose

$$-\infty < \alpha = \inf\ S < \operatorname{Sup} S = \beta < \infty.$$

Then S is (Lebesgue) measurable iff the characteristic function of S is GRI over $[\alpha, \beta]$ (see second sentence of Section 4), namely, iff there is a real number I so that, for every $\varepsilon > 0$, there is a positive function $\delta_\varepsilon(t)$ on $[\alpha, \beta]$ such that

$$\left| I - \sum (x_k - x_{k-1}) \right| < \varepsilon,$$

whenever $\alpha = x_0 < x_1 < \cdots < x_n = \beta$, $x_{k-1} \le t_k \le x_k$, $x_k - x_{k-1} < \delta_\varepsilon(t_k)$ for $k = 1, 2, \ldots, n$, where the last sum is extended over those $k = 1, 2, \ldots, n$ for which $t_k \in S$ (and an "empty" $\sum$ means 0). In that case, such an I is unique and is called the (Lebesgue) measure of S.

SECTION 8

In Definition 6, the requirement $x_k - x_{k-1} < \delta_\varepsilon(t_k)$ for $k = 1, 2, \ldots, n$ may clearly be replaced by $|x_{k-1} - t_k| < \delta_\varepsilon(t_k)$, $|x_k - t_k| < \delta_\varepsilon(t_k)$, $k = 1, 2, \ldots, n$. Suppose we make such a replacement but also drop the requirement $x_{k-1} \le t_k \le x_k$, $k = 1, 2, \ldots, n$. Then if a corresponding I exists, it is clearly (GRI)$\int_a^b f(x)\,dx$. It turns out [24,25] that such an I exists iff f is summable over $[a, b]$ in which case

$$I = (L) \int_a^b f(x)\,dx.$$

We have thus completed a full circle: we started and ended with the Lebesgue integral. During this tour we encountered several facets of GRI which is stronger than the Lebesgue integral which it should replace.

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